CURL OF A VECTOR

The **curl** of **A** is an axial (or rotational) vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.[2]

In Cartesian coordinates the curl of **A** is easily found using

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

or

$$\boxed{\begin{aligned}\nabla \times \mathbf{A} = {} & \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\mathbf{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\mathbf{a}_y \\ & + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\mathbf{a}_z\end{aligned}}$$

the curl of $\mathbf{A}$ in cylindrical coordinates as

$$\nabla \times \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} \mathbf{a}_\rho & \rho\, \mathbf{a}_\phi & \mathbf{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

or

$$\begin{aligned} \nabla \times \mathbf{A} = {} & \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right] \mathbf{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] \mathbf{a}_\phi \\ & + \frac{1}{\rho}\left[\frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right] \mathbf{a}_z \end{aligned}$$

and in spherical coordinates as

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & r\, \mathbf{a}_\theta & r \sin\theta\, \mathbf{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin\theta\, A_\phi \end{vmatrix}$$

or

$$\begin{aligned} \nabla \times \mathbf{A} = {} & \frac{1}{r \sin\theta}\left[\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right] \mathbf{a}_r \\ & + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial (r A_\phi)}{\partial r}\right] \mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right] \mathbf{a}_\phi \end{aligned}$$

EXAMPLE

Determine the curl of these vector fields:

(a) $\mathbf{P} = x^2yz\,\mathbf{a}_x + xz\,\mathbf{a}_z$

(b) $\mathbf{Q} = \rho \sin \phi\,\mathbf{a}_\rho + \rho^2 z\,\mathbf{a}_\phi + z \cos \phi\,\mathbf{a}_z$

(c) $\mathbf{T} = \dfrac{1}{r^2} \cos \theta\,\mathbf{a}_r + r \sin \theta \cos \phi\,\mathbf{a}_\theta + \cos \theta\,\mathbf{a}_\phi$

Solution:

$$\text{(a)}\quad \nabla \times \mathbf{P} = \left(\frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial P_x}{\partial z} - \frac{\partial P_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y}\right)\mathbf{a}_z$$

$$= (0 - 0)\,\mathbf{a}_x + (x^2y - z)\,\mathbf{a}_y + (0 - x^2z)\,\mathbf{a}_z$$

$$= (x^2y - z)\mathbf{a}_y - x^2z\mathbf{a}_z$$

$$\text{(b)}\quad \nabla \times \mathbf{Q} = \left[\frac{1}{\rho}\frac{\partial Q_z}{\partial \phi} - \frac{\partial Q_\phi}{\partial z}\right]\mathbf{a}_\rho + \left[\frac{\partial Q_\rho}{\partial z} - \frac{\partial Q_z}{\partial \rho}\right]\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho Q_\phi) - \frac{\partial Q_\rho}{\partial \phi}\right]\mathbf{a}_z$$

$$= \left(\frac{-z}{\rho}\sin \phi - \rho^2\right)\mathbf{a}_\rho + (0 - 0)\mathbf{a}_\phi + \frac{1}{\rho}(3\rho^2 z - \rho \cos \phi)\mathbf{a}_z$$

$$= -\frac{1}{\rho}(z \sin \phi + \rho^3)\mathbf{a}_\rho + (3\rho z - \cos \phi)\mathbf{a}_z$$

$$
\begin{aligned}
\text{(c)}\quad \nabla \times \mathbf{T} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(T_\phi \sin\theta) - \frac{\partial}{\partial\phi}T_\theta\right]\mathbf{a}_r \\
&\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}T_r - \frac{\partial}{\partial r}(rT_\phi)\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rT_\theta) - \frac{\partial}{\partial\theta}T_r\right]\mathbf{a}_\phi \\
&= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\cos\theta\sin\theta) - \frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)\right]\mathbf{a}_r \\
&\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\frac{(\cos\theta)}{r^2} - \frac{\partial}{\partial r}(r\cos\theta)\right]\mathbf{a}_\theta \\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\sin\theta\cos\phi) - \frac{\partial}{\partial\theta}\frac{(\cos\theta)}{r^2}\right]\mathbf{a}_\phi \\
&= \frac{1}{r\sin\theta}(\cos 2\theta + r\sin\theta\sin\phi)\mathbf{a}_r + \frac{1}{r}(0 - \cos\theta)\mathbf{a}_\theta \\
&\quad + \frac{1}{r}\left(2r\sin\theta\cos\phi + \frac{\sin\theta}{r^2}\right)\mathbf{a}_\phi \\
&= \left(\frac{\cos 2\theta}{r\sin\theta} + \sin\phi\right)\mathbf{a}_r - \frac{\cos\theta}{r}\mathbf{a}_\theta + \left(2\cos\phi + \frac{1}{r^3}\right)\sin\theta\,\mathbf{a}_\phi
\end{aligned}
$$

LAPLACIAN OF A SCALAR

The **Laplacian** of a scalar field *V*, written as $\nabla^2 V$, is the divergence of the gradient of *V*.

Thus, in Cartesian coordinates,

$$\text{Laplacian } V = \nabla \cdot \nabla V = \nabla^2 V$$

$$= \left[\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right] \cdot \left[\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right]$$

that is,

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

In cylindrical coordinates,

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}}$$

and in spherical coordinates,

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}}$$

EXAMPLE

Find the Laplacian of the scalar fields

(a) $V = e^{-z} \sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10r \sin^2 \theta \cos \phi$

Solution:

The Laplacian in the Cartesian system can be found by taking the first derivative and later the second derivative.

(a) $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(2e^{-z} \cos 2x \cosh y) + \frac{\partial}{\partial y}(e^{-z} \cos 2x \sinh y)$$

$$+ \frac{\partial}{\partial z}(-e^{-z} \sin 2x \cosh y)$$

$$= -4e^{-z} \sin 2x \cosh y + e^{-z} \sin 2x \cosh y + e^{-z} \sin 2x \cosh y$$

$$= -2e^{-z} \sin 2x \cosh y$$

$$\text{(b)}\quad \nabla^2 U = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(2\rho^2 z\cos 2\phi) - \frac{1}{\rho^2}4\rho^2 z\cos 2\phi + 0$$

$$= 4z\cos 2\phi - 4z\cos 2\phi$$

$$= 0$$

$$\text{(c)}\quad \nabla^2 W = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial W}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial W}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 W}{\partial \phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(10r^2\sin^2\theta\cos\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(10r\sin 2\theta\sin\theta\cos\phi)$$

$$- \frac{10r\sin^2\theta\cos\phi}{r^2\sin^2\theta}$$

$$= \frac{20\sin^2\theta\cos\phi}{r} + \frac{20r\cos 2\theta\sin\theta\cos\phi}{r^2\sin\theta}$$

$$+ \frac{10r\sin 2\theta\cos\theta\cos\phi}{r^2\sin\theta} - \frac{10\cos\phi}{r}$$

$$= \frac{10\cos\phi}{r}(2\sin^2\theta + 2\cos 2\theta + 2\cos^2\theta - 1)$$

$$= \frac{10\cos\phi}{r}(1 + 2\cos 2\theta)$$

Stokes's theorem

$$\boxed{\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}}$$

Divergence theorem

$$\boxed{\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A}\, dv}$$